EPRL/FK Group Field Theory

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Abstract

The purpose of this short note is to clarify the Group Field Theory vertex and propagators corresponding to the EPRL/FK spin foam models and to detail the subtraction of leading divergences of the model.

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1 Introduction

Group field theories (GFTs) (see [1, 2, 3] and [4, 5]) are the higher dimensional generalization of random matrix models. Like in matrix models, the Feynman graphs of group field theory are dual to triangulations (gluing of simplices). The combinatorics of a Feynman graph encodes the topology of the gluing while its amplitude encodes a sum over metrics compatible with a fixed gluing. The correlation functions of GFT's sum over both metrics

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and topologies. Generically GFT's generate topological singularities [6], but the most dangerous can be eliminated by restricting the allowed gluings by a "coloring" prescription [7].

The metrics appear in GFT through their holonomies, group elements of $SO(D)$ for D dimensional manifolds. The Feynman amplitudes are therefore integrals on $SO(D)$, or sums over spin indices in Fourier space. Such amplitudes are also referred to as spin foams [8]. Due to the Wick theorem, GFT provide a prescription of the class of graphs that should be summed, together with their combinatoric weights.

In any quantum field theory there is some ambiguity in the definition of propagators and vertices. A vertex can be dressed by an arbitrary fraction of the propagator without changing the bulk theory, provided we amputate each propagator by the square of that fraction. What fixes this ambiguity in ordinary quantum field theory is a locality requirement on the vertices.

In [9], such a locality requirement was proposed for GFTs, namely to restrict their vertices to simple products of δ functions which identify group elements in strands crossing the vertex. Everything else should be considered part of the propagator. Beware that this is *not* the usual spin-foam terminology. However, as we will see in the sequel, it immediately leads to a well defined and simple prescription to identify divergences.

In dimension D the simplest and most natural vertex with this locality property represents $D+1$ subsimplices of dimension $D-1$ bounding a D dimensional simplex (hence connected through $D(D+1)/2$ such δ functions). The fields are functions on $SO(D)^D$, and the D-stranded propagators represent the gluing of D dimensional simplices along $D-1$ dimensional subsimplices.

Using as propagator a diagonal $SO(D)$ gauge averaging projection T (ensuring flatness of the holonomies), the amplitude of a Feynman graph equals the partition function of a BF theory discretized on the dual gluing of simplices. Recently such models have received increased attention and various partial power counting results have been established, either for generic three dimensional models [10, 11] or for colored and linearized models [12, 13, 14]

Gravity can be seen as a constrained version of BF theory. In line with this approach, new spin foam rules have been proposed to implement the so called Plebanski simplicity constrains and reproduce the partition function of fully fledged 4D gravity [15, 16, 17, 18]. These new models (referred to as EPRL/FK in this paper) mix the left and right part of $SO(4) \simeq SU(2) \times SU(2)$ in a novel way and give a central rôle to the Immirzi

parameter. Amplitudes of particular spin foams in the EPRL/FK models, revealing improved UV behavior, have been derived in [19] and recovered in [9].

But, as spin foams are only Feynman graphs of the GFT, one still needs to identify an appropriate GFT propagator which generates the EPRL/FK spin foam amplitudes. A first step in this direction has been performed in [9], were the propagator (written in terms of coherent states) was computed as a product of gauge (T) and simplicity (S) projection operators, $C = TST$[1]. Note that C has a non trivial spectrum, hence is suited for a RG analysis. Some steps have already been performed in [9] to write the EPRL/FK action in terms of group elements, here we propose another equivalent formulation, free of explicit sums over coherent states, and which might lead to a transparent saddle point analysis for estimating graph amplitudes.

In this paper we obtain the EPRL/FK propagator in group space and consequently more compact formulas for both the propagator and the Feynman amplitudes of the GFT underlying EPRL/FK spin foams. Our formulas are well defined for irrational values of the Immirzi parameter and constitute a better starting point for slicing the propagator according to its spectrum (and subsequently a fully fledged RG analysis). Although such an analysis is in progress [20], in this paper we limit ourselves to a non technical introduction of this GFT written exclusively in the group variables. It must be mentioned that, in a somewhat different perspective, improved GFT's [21, 22] have already been proposed to implement directly the simplicity constrains. A direct comparison of the action functionals proposed in [21, 22] and our results shows that they are in fact quite different and it is still an open question which one (if any) of these GFT's is the best suited to describe gravity.

This paper is organized as follows: section 2 details the simplicity projector S in direct space in terms of characters and section 3 presents the EPRL/FK propagator. Section 4 computes the Feynman amplitudes of arbitrary graphs, and section 5 explains the subtraction of leading divergences. As an added bonus we confirm the power counting estimate found in [19, 9] for a particular graph very effectively in our new representation. Technical details are presented in two appendices.

[1] Beware that different letters are used in [9].

2 The Simplicity Projector S

The coherent spin states [24] form an over complete basis in the $SU(2)$ representation spaces. The decomposition of an operator over an over complete basis is not unique, thus one has many possible choices for the kernel of the EPRL/FK simplicity projector S. In [17] (and subsequently in [9]), it is taken to be

$$S = \sum_{j^+,j^-} \delta_j^\gamma \; d_{j^++j^-} \int dn \; |j^+,n\rangle \otimes |j^-,n\rangle \langle j^+,n| \otimes \langle j^-,n| \; , \tag{1}$$

with

$$\begin{aligned} & d_{j^++j^-} = 2(j^+ + j^-) + 1 \; , \\ & \delta_j^\gamma = \delta_{\frac{j^+}{j^-} = \frac{1+\gamma}{|1-\gamma|}} = \delta_{|1-\gamma| j^+ = (1+\gamma) j^-} \; . \end{aligned} \tag{2}$$

This is a perfectly valid choice but it has one major drawback. Although, as S is a projector, $S^2 = S$, the square of eq. (1) is

$$\begin{aligned} & S = S^2 = \sum_{j^+,j^-} \delta_j^\gamma \; d^2_{j^++j^-} \int dn dn' \\ & |j^+,n\rangle \otimes |j^-,n\rangle \langle j^+ + j^-, n | j^+ + j^-, n' \rangle \langle j^+,n'| \otimes \langle j^-,n'| \; , \end{aligned} \tag{3}$$

which looks quite different. This discrepancy is explained by the over completeness of the coherent states basis[2]. It the sequel, we choose the representation provided in eq. (3) as it is better suited for explicit computations.

Remark that the δ_j^γ does not really make sense (e.g. if γ is irrational) but should be understood in an asymptotic sense as $j_\pm \to \infty$. This will be detailed later, and the formulas we will derive for the amplitudes of the theory ultimately make sense for any γ.

It is important to realize that the eq. (3) is in fact only a shortened (and somewhat confusing) notation. The operator S acts on functions defined on $SO(4)$ which decompose in Fourier modes as

$$f(g) = \sum d_j f^j_{pm} D^j_{pm}(g) \; , \tag{4}$$

hence the *matrix* elements $D^j_{pm}(g)$ (and not the vectors $|jm\rangle$) are the analog of the plane waves. Matrix elements of the operator S therefore join a $D^{j_1}_{p_1 m_1}(g_1)$

[2] In order to conclude that S is a projector, in [9] one proves that $S^3 = S^2$, rather than proving $S^2 = S$.

to a $D^{j_2}_{p_2m_2}(g_2)$, hence have two groups of indices j_1, p_1, m_1 and j_2, p_2, m_2. To make matters worse, over $SO(4) \simeq SU(2) \times SU(2)$ each of the above six indices is in fact a double index, $\vec{j}_1 = (j_1^+, j_1^-)$, corresponding respectively to each of the two copies of $SU(2)$. In full detail S writes

$$
\begin{aligned}
& S^{\vec{j}_1,\vec{j}_2}_{(\vec{p}_1,\vec{m}_1);(\vec{p}_2,\vec{m}_2)} = \delta_{\vec{j}_1,\vec{j}_2}\delta^{\gamma}_{\vec{j}_1}\ d^2_{j_1^+ + j_1^-}\delta_{\vec{p}_1,\vec{p}_2} \\
& \int dndn'\ \langle \vec{j}_1, \vec{m}_1 | \Big(|j_1^+, n\rangle \otimes |j_1^-, n\rangle \Big) \langle j_1^+ + j_1^-, n | j_1^+ + j_1^-, n' \rangle \\
& \Big(\langle j_1^+, n' | \otimes \langle j_1^-, n' | \Big) | \vec{j}_2, \vec{m}_2 \rangle\ ,
\end{aligned} \tag{5}
$$

where $|\vec{j}, \vec{m}\rangle = |j^+, m^+\rangle \otimes |j^-, m^-\rangle$. Denoting the matrix elements of unitary representations of $SU(2) \times SU(2)$ as

$$
D^{\vec{j}}_{\vec{p}\vec{m}}(g) := D^{j^+}_{p^+m^+}(g^+) D^{j^-}_{p^-m^-}(g^-)\ , \tag{6}
$$

we find in the direct (group) space

$$
S(g_1, g_2) = \sum d_{j_1^+} d_{j_1^-} S^{\vec{j}_1\vec{j}_2}_{(\vec{p}_1\vec{m}_1);(\vec{p}_2\vec{m}_2)} D^{\vec{j}_1}_{\vec{p}_1\vec{m}_1}(g_1) \overline{D^{\vec{j}_2}_{\vec{p}_2\vec{m}_2}(g_2)}\ . \tag{7}
$$

Substituting eq. (5) yields

$$
\begin{aligned}
& S(g_1, g_2) = \sum d_{j_1^+} d_{j_1^-} D^{\vec{j}_1}_{\vec{p}_1\vec{m}_1}(g_1) \overline{D^{\vec{j}_2}_{\vec{p}_2\vec{m}_2}(g_2)} \delta_{\vec{j}_1,\vec{j}_2}\delta^{\gamma}_{\vec{j}_1}\ d^2_{j_1^+ + j_1^-}\delta_{\vec{p}_1,\vec{p}_2} \\
& \int dndn'\ \langle \vec{j}_1, \vec{m}_1 | \Big(|j_1^+, n\rangle \otimes |j_1^-, n\rangle \Big) \langle j_1^+ + j_1^-, n | j_1^+ + j_1^-, n' \rangle \\
& \Big(\langle j_1^+, n' | \otimes \langle j_1^-, n' | \Big) | \vec{j}_2, \vec{m}_2 \rangle\ ,
\end{aligned} \tag{8}
$$

and summing over $\vec{p}_2$ and $\vec{j}_2$ (and renaming $\vec{j}_1 = \vec{j}$), we get

$$
S(g_1, g_2) = \sum d_{j^+} d_{j^-}\ \delta^{\gamma}_{\vec{j}}\ d^2_{j^+ + j^-}\ D^{\vec{j}}_{\vec{m}_2\vec{m}_1}\big((g_2)^{-1} g_1\big) \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2)\ , \tag{9}
$$

with

$$
\begin{aligned}
\mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) \quad & = \int dndk\ \langle \vec{j}, \vec{m}_1 | \Big(|j^+, n\rangle \otimes |j^-, n\rangle \Big) \\
& \langle j^+ + j^-, n | j^+ + j^-, k \rangle \Big(\langle j^+, k | \otimes \langle j^-, k | \Big) | \vec{j}, \vec{m}_2 \rangle\ .
\end{aligned} \tag{10}
$$

The integral $\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)$ is evaluated in Appendix B. Substituting eq. (B.11) yields

$$\begin{aligned} S(g_1,g_2) \quad &= \sum_{j^+,j^-} d_{j^+}d_{j^-}d_{j^++j^-}\;\delta_j^\gamma \\ &\sum_{\vec{m}_1,\vec{m}_2} D^{j^+}_{m_2^+,m_1^+}\big((g_2^+)^{-1}g_1^+\big)\;D^{j^-}_{m_2^-,m_1^-}\big((g_2^-)^{-1}g_1^-\big) \\ &\sum_r \int dh\; D^{j^+}_{m_1^+m_2^+}(h)\;D^{j^-}_{m_1^-m_2^-}(h)\;D^{j^++j^-}_{-r-r}(h)\;, \end{aligned} \tag{11}$$

where the integral in the last line is performed over only one group element $h\in SU(2)$. Note that due to the selection rules the sum over r is in fact restricted to a single term $r=m_1^++m_1^-=m_2^++m_2^-$. However, although this remark is important in a detailed slice analysis of S (hence of the propagator C) we ignore it throughout this paper. The rationale behind this is that, as will be clear in the sequel, allowing this fake sum to survive yields canonical expressions in terms of the familiar $SU(2)$ group characters for all relevant quantities. Summing over $\vec{m}_1,\vec{m}_2$ and r in eq. (11) yields the compact expression

$$\begin{aligned} S(g_1,g_2) \;&=\; \sum d_{j^+}d_{j^-}d_{j^++j^-}\;\delta_j^\gamma \\ &\quad \int dh\,\chi^{j^+}\big((g_2^+)^{-1}g_1^+h\big)\,\chi^{j^-}\big((g_2^-)^{-1}g_1^-h\big)\,\chi^{j^++j^-}(h) \\ &=\; \sum d_{j^+}d_{j^-}\;d_J\;\delta_j^\gamma\delta_{J=j^++j^-} \\ &\quad \int dh\,\chi^{j^+}\big(g_1^+h(g_2^+)^{-1}\big)\,\chi^{j^-}\big(g_1^-h(g_2^-)^{-1}\big)\,\chi^{J}(h)\;, \end{aligned} \tag{12}$$

with $\chi^j(g)=\mathrm{Tr}_j(g)=\sum_k D^j_{kk}(g)$ the character of g in the representation j. Note that at this stage all the coherent state integrals have been performed, and S is written exclusively in terms of group integrals and characters. Using $\overline{\chi(h)}=\chi(h^\dagger)$ and the orthogonality of characters

$$\int dp\;\chi^j(ap^{-1})\chi^{j'}(pb)=\frac{1}{d_j}\delta_{jj'}\chi^j(ab)\;, \tag{13}$$

one can check directly using eq. (12) that S is a projector. Note that eq. (12) makes sense for any value of γ as the character of a group element $\chi^j(g)=\frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$ is well defined for all values of j, half integer or not.

The simplicity projector S admits several limiting cases

- $\gamma = 1$ sets $j^- = 0$, and S becomes

$$
\begin{aligned}
S(g_1, g_2) &= \sum_{j^+, J} d_{j^+} d_J \delta_{J=j^+} \int dh\, \chi^{j^+}\big(g_1^+ h (g_2^+)^{-1}\big) \chi^J(h) \\
&= \sum_J d_J \chi^J(g_1^+ (g_2^+)^{-1}) \;=\; \delta(g_1^+ (g_2^+)^{-1}) \;, \qquad (14)
\end{aligned}
$$

  leading to a BF theory for the + copy of $SU(2)$.

- Ignoring both δ_j^γ $\delta_{J=j^+ + j^-}$ yields

$$
\begin{aligned}
S(g_1, g_2) &= \sum_{j^+, j^-, J} d_{j^+} d_{j^-} d_J \int dh\, \chi^{j^+}\big(g_1^+ h (g_2^+)^{-1}\big) \\
&\quad\; \chi^{j^-}\big(g_1^- h (g_2^-)^{-1}\big)\, \chi^J(h) \\
&= \int dh\, \delta\big(g_1^+ h (g_2^+)^{-1}\big)\, \delta\big(g_1^- h (g_2^-)^{-1}\big)\, \delta(h) \\
&= \delta\big(g_1^+ (g_2^+)^{-1}\big)\, \delta\big(g_1^- (g_2^-)^{-1}\big) \;, \qquad (15)
\end{aligned}
$$

  which is the $SO(4)$ BF theory.

- $\gamma \to \infty$ leads to $j^+ = j^-$ and

$$
S(g_1, g_2) = \sum_j d_j^2\, d_{2j} \int dh\, \chi^j\big(g_1^+ h (g_2^+)^{-1}\big)\, \chi^j\big(g_1^- h (g_2^-)^{-1}\big)\, \chi^{2j}(h) \;, \qquad (16)
$$

  which is the Barrett Crane spin foam model [23].

Returning to eq. (12), note that S admits a single sum representation

$$
\begin{aligned}
S(g_1, g_2) &= \sum_J \Big[J(1-\gamma) + 1\Big] \Big[J(1+\gamma) + 1\Big] \Big[2J + 1\Big] \\
&\quad\; \int dh\, \chi^{J\frac{1+\gamma}{2}}\big(g_1^+ h (g_2^+)^{-1}\big)\, \chi^{J\frac{1-\gamma}{2}}\big(g_1^- h (g_2^-)^{-1}\big)\, \chi^J(h) \;. \qquad (17)
\end{aligned}
$$

3 The EPRL/FK propagator

In four dimensions the GFT lines have four strands. To build the EPRL/FK propagator one needs to compose four simplicity projectors, one for each strand, with two gauge invariance projectors, common to all four strands.

The ordinary $SO(4)$ gauge invariance propagator, T, corresponding to left invariant fields under the diagonal group action on their arguments, i.e. fields satisfying

$$\phi(g_1 h, g_2 h, g_3 h, g_4 h) = \phi(g_1, g_2, g_3, g_4) \ , \tag{18}$$

has kernel

$$T(\{g_s\}, \{g'_s\}) = \int dh^+ dh^- \prod_{s=1}^{4} \delta\big(g_s^+ h^+ (g_s'^+)^{-1}\big) \delta\big(g_s^- h^- (g_s'^-)^{-1}\big) \ , \tag{19}$$

where $\{g_s\}$ denotes a collection of four group elements associated to the strands. The pair of integration variables (h^+, h^-) is common to all four strands of a line. The propagator writes

$$\begin{aligned} C(\{g_s\};\{g'_s\}) \ &= \ \int \prod_s \big(du_s dv_s\big) \ T(\{g_s\}, \{u_s\}) \\ &\quad \Big(\prod_s S(u_s, v_s)\Big) \ T(\{v_s\}, \{g'_s\}) \ , \end{aligned} \tag{20}$$

or in detail, denoting $\delta_J = \delta_{J=j^+ + j^-}$ and $h^{\pm}_{\text{in}}, h^{\pm}_{\text{out}}$ the dummy variables corresponding to the two T operators

$$\begin{aligned} & C(\{g_s\};\{g'_s\}) = \sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} d_{J_s} \, \delta^{\gamma}_{j_s} \ \delta_{J_s} \int dh^{\pm}_{\text{in}} dh^{\pm}_{\text{out}} \int \prod_s dh_s \\ & \int \prod_s \big(du_s^{\pm} dv_s^{\pm}\big) \ \prod_s \delta\big(\, g_s^+ \, h_{\text{in}}^+ \, (u_s^+)^{-1}\big) \delta\big(g_s^- \, h_{\text{in}}^- \, (u_s^-)^{-1}\big) \\ & \prod_s \chi^{j_s^+} (u_s^+ \, h_s \, (v_s^+)^{-1}) \, \chi^{j_s^-} (u_s^- \, h_s \, (v_s^-)^{-1}) \, \chi^{J_s}(h_s) \\ & \prod_s \delta\big(v_s^+ \, h_{\text{out}}^+ \, (g_s'^+)^{-1}\big) \delta\big(v_s^- \, h_{\text{out}}^- \, (g_s'^-)^{-1}\big) \ , \end{aligned} \tag{21}$$

and integrating over $u_s^{\pm}, v_s^{\pm}$ we get

$$\begin{aligned} & C(\{g_s\};\{g'_s\}) = \sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} d_{J_s} \, \delta^{\gamma}_{j_s} \ \delta_{J_s} \int dh^{\pm}_{\text{in}} dh^{\pm}_{\text{out}} \int \prod_s dh_s \\ & \prod_s \chi^{j_s^+} (g_s^+ \, h_{\text{in}}^+ \, h_s \, h_{\text{out}}^+ \, (g_s'^+)^{-1}) \, \chi^{j_s^-} (g_s^- \, h_{\text{in}}^- \, h_s \, h_{\text{out}}^- \, (g_s'^-)^{-1}) \, \chi^{J_s}(h_s) \ . \end{aligned} \tag{22}$$

A EPRL/FK group field theory line is represented together with all its associated group elements in figure 1.

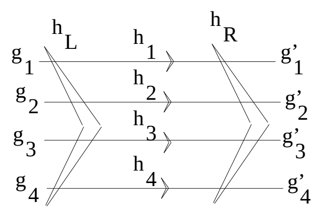

Figure 1: A EPRL/FK line.

4 Feynman amplitudes

A Feynman graph of the EPRL/FK group field theory is made of propagators (eq. (22)) and vertices made of trivial conservation δ functions. Note that the strands are mixed in eq. (22) only through the variables $h^{\pm}_{\rm in,out}$. In particular, when composing several such propagators, all the intermediate g_s integrals are factored according to the strands. When computing the amplitude of a graph, the integrand is therefore factored into contributions associated either to closed strands (called faces) or to open (external) strands.

The composition of two successive strand contributions writes

$$\begin{aligned}
&\sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} d_{J_s} \, \delta^{\gamma}_{j_s} \; \delta_{J_s} \sum_{j_{s'}^+, j_{s'}^-, J_{s'}} d_{j_{s'}^+} d_{j_{s'}^-} d_{J_{s'}} \, \delta^{\gamma}_{j_{s'}} \; \delta_{J_{s'}} \int dg'^{\pm}_s \\
&\chi^{j_s^+}(g^+_s \, h^+_{\rm in} \, h_s \, h^+_{\rm out} \, (g'^+_s)^{-1}) \, \chi^{j_s^-}(g^-_s \, h^-_{\rm in} \, h_s \, h^-_{\rm out} \, (g'^-_s)^{-1}) \, \chi^{J_s}(h_s) \\
&\chi^{j_{s'}^+}(g'^+_s \, p^+_{\rm in} \, p_s \, p^+_{\rm out} \, (g''^+_s)^{-1}) \, \chi^{j_{s'}^-}(g'^-_s \, p^-_{\rm in} \, p_s \, p^-_{\rm out} \, (g''^-_s)^{-1}) \, \chi^{J_{s'}}(p_s) \; , \quad (23)
\end{aligned}$$

which, using the orthogonality of characters eq. (13) computes to

$$\begin{aligned}
&\sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} (d_{J_s})^2 \, \delta^{\gamma}_{j_s} \; \delta_{J_s} \\
&\chi^{j_s^+}(g^+_s \, h^+_{\rm in} \, h_s \, h^+_{\rm out} \, (p^+_{\rm in} \, p_s \, p^+_{\rm out}) \, (g''^+_s)^{-1}) \\
&\chi^{j_s^-}(g^-_s \, h^-_{\rm in} \, h_s \, h^-_{\rm out} \, (p^-_{\rm in} \, p_s \, p^-_{\rm out}) \, (g''^-_s)^{-1}) \\
&\chi^{J_s}(h_s) \chi^{J_s}(p_s) \; . \quad (24)
\end{aligned}$$

In an arbitrary Feynman amplitude we therefore have one surviving independent sum per face of the graph and one per external strand.

To write the full amplitude of a graph $\mathcal{G}$ we introduce some notations. We denote the two couples of in and out variables of a line l by $h^{\pm}_{{\rm in};l}$ and $h^{\pm}_{{\rm out};l}$. We denote ∂f the set of lines of the boundary of the face f and $|\partial f|$

its cardinal. For each line $l \in \partial f$ we have a variable h_{lf} (corresponding to h_s in eq. (22)). Furthermore, we denote ϵ_{lf} the incidence matrix of lines within faces [9, 13], which is 0 if $l \notin \partial f$ and 1 (or -1) if $l \in \partial f$ and the orientations of l and f coincide (or not). Finally, denoting $\mathcal{L}_{\mathcal{G}}$ the set of lines and $\mathcal{F}_{\mathcal{G}}$ the set of faces of $\mathcal{G}$, the amplitude writes

$$\begin{aligned} A_{\mathcal{G}}(\{g_s^+\},\{g_s^-\}) = &\sum_{j_f^+, j_f^-, J_{lf}} \Big(\prod_{f \in \mathcal{F}_{\mathcal{G}}} d_{j_f^+} d_{j_f^-} \, \delta^{\gamma}_{j_f} \, \big(\prod_{l \in \partial f} d_{J_{lf}} \delta_{J_{lf} = j_f^+ + j_f^-} \big) \Big) \qquad (25) \\ &\int \Big[\prod_{l \in \mathcal{L}_{\mathcal{G}}} dh^{\pm}_{\mathrm{in};\,l} dh^{\pm}_{\mathrm{out};\,l} \Big] \int \Big[\prod_{\substack{l \in \mathcal{L}_{\mathcal{G}}, f \in \mathcal{F}_{\mathcal{G}} \\ l \in \partial f}} dh_{lf} \Big] \Big[\prod_{\substack{l \in \mathcal{L}_{\mathcal{G}}, f \in \mathcal{F}_{\mathcal{G}} \\ l \in \partial f}} \chi^{J_{lf}}(h_{lf}) \Big] \\ &\prod_{f \in \mathcal{F}_{\mathcal{G}}} \Big[\chi^{j_f^+} \Big(\prod_{l \in \partial f} (h^+_{\mathrm{in};\,l}\, h_{lf}\, h^+_{\mathrm{out};\,l})^{\epsilon_{lf}} \Big) \, \chi^{j_f^-} \Big(\prod_{l \in \partial f} (h^-_{\mathrm{in};\,l}\, h_{lf}\, h^-_{\mathrm{out};\,l})^{\epsilon_{lf}} \Big) \Big] \,, \end{aligned}$$

where for external, open faces, with group elements at the endpoints $g_s^{\pm}$ and $g_s'^{\pm}$, the last line is replace by

$$\begin{aligned} &\chi^{j_f^+} \Big[(g_s^+)^{\epsilon_{ef}} \Big(\prod_{l \in \partial f} (h^+_{\mathrm{in};\,l}\, h_{lf}\, h^+_{\mathrm{out};\,l})^{\epsilon_{lf}} \Big) (g_s'^+)^{\epsilon_{ef}} \Big] \\ &\chi^{j_f^-} \Big[(g_s^-)^{\epsilon_{ef}} \Big(\prod_{l \in \partial f} (h^-_{\mathrm{in};\,l}\, h_{lf}\, h^-_{\mathrm{out};\,l})^{\epsilon_{lf}} \Big) (g_s'^-)^{\epsilon_{ef}} \Big] \,, \end{aligned} \qquad (26)$$

with ϵ_{ef} the incidence matrix of external points with faces.

Before concluding this section lest us note that one could use an arbitrary power of the TST operator as propagator, since any power would effectively implement the Plebanski constraints. We can presumably, in this category of theories generalizing EPRL/FK, always adjust the power k so as to find a just renormalizable theory[3]. The amplitudes of such a model will have the same form as eq. (25), but with extra insertions of intermediate variables $h^{\pm}_{\mathrm{in,out}}$ and h_{lf} along the faces improving the power counting of the theory.

5 Subtraction, locality, and all that

Starting form the eq. (25) of the Feynman amplitude of a graph one can address the subtraction of divergences in this theory. This procedure is straightforward with the definition of locality proposed in [9].

[3]We acknowledge D. Oriti for making this remark during a most enjoyable conference in AEI, Golm.

Take the example of a two point function. The amplitude of a connected graph writes in terms of the amplitude of the amputated graph as

$$A_{\mathcal{G}}(\phi) = \int dg_s dg'_s \; \phi(\{g_s\}) \; \phi(\{g'_s\}) \; A_{\mathcal{G}}(\{g_s\}, \{g'_s\}) \; , \tag{27}$$

the leading ("mass") divergence is immediately identified by Taylor developing "at zeroth order" the field $\phi(\{g'_s\})$[4] around $\{g'_s\} = \{g_s\}$

$$\begin{aligned} A_{\mathcal{G}}(\phi) \;\; &= \;\; \int dg_s \; \phi(\{g_s\}) \; \phi(\{g_s\}) \int dg'_s \; A_{\mathcal{G}}(\{g_s\}, \{g'_s\}) + \text{ sub leading} \\ &= \delta\mu_{\mathcal{G}} \int dg_s \; \phi(\{g_s\}) \; \phi(\{g_s\}) + \text{sub leading} \; . \end{aligned} \tag{28}$$

Taking into account eq. (26), we note that the integration over the external field $g'^{\pm}_s$ fixes all j^+_f, j^-_f and J_{lf} to 0 and the external strand contribution drops out of eq. (25). In general, the leading divergence of any graph $\mathcal{G}$ is therefore obtained by integrating eq. (25) ignoring the external strands.

Take the example of the graph $\mathcal{G}$ drawn schematically in figure 2. All lines have parallel strands, and are oriented from left to right. We denote the lines 1 to 4 (which can be interpreted as colors in a colored model), and the face by the couple of labels of the lines composing them. The set of internal faces of this graph is therefore $f = \{f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}$.

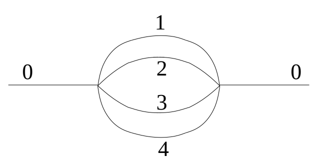

Figure 2: A graph exhibiting a mass divergence.

The mass divergence of $\mathcal{G}$ writes

$$\begin{aligned} \delta\mu_{\mathcal{G}} \;\; = \;\; &\sum_{j^+_{12}, j^-_{12}, J_{1,12}, J_{2,12}, \dots} \Big(d_{j^+_{12}} d_{j^-_{12}} \delta^{\gamma}_{j_{12}} \big(d_{J_{1,12}} \delta_{J_{1,12}} d_{J_{2,12}} \delta_{J_{2,12}} \big) \Big) \dots \\ &\int dh^{\pm}_{\text{in},1} dh^{\pm}_{\text{out},1} \dots \int dh_{1,12} dh_{2,12} \dots \chi^{J_{1,12}}(h_{1,12}) \chi^{J_{2,12}}(h_{2,12}) \dots \end{aligned}$$

[4]As always, sub leading divergences are more difficult to extract (one needs to push further the Taylor development of the external fields), and is deferred for further work.

$$\chi^{j^+_{12}}(h^+_{\text{in},1}h_{1,12}h^+_{\text{out},1}(h^+_{\text{in},2}h_{2,12}h^+_{\text{out},2})^{-1})$$
$$\chi^{j^-_{12}}(h^-_{\text{in},1}h_{1,12}h^-_{\text{out},1}(h^-_{\text{in},2}h_{2,12}h^-_{\text{out},2})^{-1})\ldots \ . \qquad (29)$$

In eq. (29) we have 6 independent sums, 16 integrals over line $h^{\pm}_{\text{in,out}}$ variables, 12 integrations over $h_{i,ij}$ strand variables of a product of 24 characters. Unsurprisingly, the full evaluation of the amplitude of this graph is somewhat involved (see [19, 9]), but fortunately one can derive its degree of divergence in our group representation relatively straight forward.

Divergences arise for large values of the spin labels $j^{\pm}, J$, thus we cutoff all the sums by some sharp cutoff Λ. Each $d^{\pm}_j, d_J$ factor in the first line of eq. (29) will bring a factor Λ. The integrals over the characters are of the form

$$\int \prod_{j=1}^{n} d\theta_j \Big(\sin\frac{\theta_j}{2}\Big)^2 \int_{S^2} dq_j \ F(\Lambda,\theta_j,q_j) \ , \qquad (30)$$

where we have used the representation eq. (A.1) of the Haar measure over $SU(2)$. The integrals over the normals q_j are bounded by 1 and will be ignored. The integrals over θ_j will be evaluated by some saddle point approximation. The saddle point equations are $\theta_j = \theta^s_j$ with

$$\theta^s_p = 0 \quad \forall p \le k \ , \qquad \theta^s_p \neq 0 \quad \forall p > k \ . \qquad (31)$$

The behavior of eq. (30) is strongly dependent of k. In fact, when translating at the saddle point $x_j = \theta_j - \theta^s_j$, eq. (30) writes

$$\int \prod_{j=1}^{k} \Big(\sin\frac{x_j}{2}\Big)^2 dx_j \prod_{j=k+1}^{n} \Big(\sin\frac{x_j+\theta^s_j}{2}\Big)^2 dx_j F(\Lambda, x+\theta^s) \ , \qquad (32)$$

and performing the rescaling $x_j = \frac{u_j}{\sqrt{\Lambda}}$ close to the saddle point we get

$$\int \prod_{j=1}^{k} \Big(\sin\frac{u_j}{2\sqrt{\Lambda}}\Big)^2 \frac{du_j}{\sqrt{\Lambda}} \prod_{j=k+1}^{n} \Big(\sin\frac{\frac{u_j}{\sqrt{\Lambda}}+\theta^s_j}{2}\Big)^2 \frac{du_j}{\sqrt{\Lambda}} F(\Lambda, \frac{u}{\sqrt{\Lambda}}+\theta^s)$$
$$\approx \frac{1}{(\sqrt{\Lambda})^{3k}} \frac{1}{(\sqrt{\Lambda})^{n-k}}$$
$$\int \prod_{j=1}^{k} \frac{u_j^2}{4} du_j \prod_{j=k+1}^{n} \Big(\sin\frac{\theta^s_j}{2}\Big)^2 du_j \ F(\Lambda, \frac{u}{\sqrt{\Lambda}}+\theta^s) \ , \qquad (33)$$

and the remaining integral gives no extra scaling in Λ. Therefore the scaling of eq. (30) is fixed by n (the number of integration variables) and k (the number of directions with saddle point equation $\theta_j = 0$).

For the graph of figure 2, we change variables to

$$\begin{aligned}
(\tilde{h}^+_{in;2})^{-1} &= h^+_{in,1} h_{1,12} h^+_{out,1} (h^+_{out,2})^{-1} h^{-1}_{2,12} (h^+_{in,2})^{-1} \\
(\tilde{h}^+_{in;3})^{-1} &= h^+_{in,1} h_{1,13} h^+_{out,1} (h^+_{out,3})^{-1} h^{-1}_{3,13} (h^+_{in,3})^{-1} \\
(\tilde{h}^+_{in;4})^{-1} &= h^+_{in,1} h_{1,14} h^+_{out,1} (h^+_{out,4})^{-1} h^{-1}_{4,14} (h^+_{in,4})^{-1} \ ,
\end{aligned} \tag{34}$$

and similarly for the $-$ variables. This brings the contribution of the faces f_{12}, f_{13}, f_{14} into the form

$$\begin{aligned}
&\chi^{j^+_{12}}\big((\tilde{h}^+_{in;2})^{-1}\big)\chi^{j^+_{13}}\big((\tilde{h}^+_{in;3})^{-1}\big)\chi^{j^+_{14}}\big((\tilde{h}^+_{in;4})^{-1}\big) \\
&\chi^{j^-_{12}}\big((\tilde{h}^-_{in;2})^{-1}\big)\chi^{j^-_{13}}\big((\tilde{h}^-_{in;3})^{-1}\big)\chi^{j^-_{14}}\big((\tilde{h}^-_{in;4})^{-1}\big) \ ,
\end{aligned} \tag{35}$$

while the (+ part) contribution of the face f_{23} becomes

$$\begin{aligned}
&\tilde{h}^+_{in;2} \ h^+_{in,1} h_{1,12} h^+_{out,1} (h^+_{out,2})^{-1} h^{-1}_{2,12} \\
&h_{2,23} h^+_{out,2} (h^+_{out,3})^{-1} h^{-1}_{3,23} \\
&h_{3,13} h^+_{out,3} (h^+_{out,1})^{-1} h^{-1}_{1,13} (h^+_{in,1})^{-1} (\tilde{h}^+_{in;3})^{-1} \ ,
\end{aligned} \tag{36}$$

and similarly for the faces f_{24}, f_{24} and f_{34}. Note that all the remaining variables, ($h^+_{\text{in;1}}$ and $h^+_{\text{out;1}}$, $h^+_{\text{out;2}}$, $h^+_{\text{out;3}}$, $h^+_{\text{out;4}}$) appear always in pairs h, h^{-1}.

The integration variables h_{lf} and $\tilde{h}$ appear explicitly as arguments of some character

$$\int dh \ \chi^j(h) F(h, \dots) \ . \tag{37}$$

For all this variables, and the associated $\theta^s_h \neq 0$ as

$$\int dh \ \chi^j(h) F(h) = \int d\theta_h \ \sin\frac{\theta_h}{2} \ \sin\frac{(2j+1)\theta_h}{2} \ F(\theta_h, \dots) \ , \tag{38}$$

and the integrand is exactly zero at $\theta_h = 0$. It is easy to check that the remaining group elements, as they appear only in pairs h, h^{-1} have $\theta_h = 0$ at the saddle. We therefore have $12 \times h_{lf} + 3 \times \tilde{h}^+ + 3 \times \tilde{h}^-$ variables with $\theta^s \neq 0$ and $5 \times h^+ + 5 \times h^-$ variables with $\theta^s = 0$. The scaling at the saddle point is, according to eq. (33),

$$\frac{1}{\sqrt{\Lambda}^{3\times 10}} \frac{1}{\sqrt{\Lambda}^{18}} = \Lambda^{-24} \ . \tag{39}$$

In eq. (29) we count 6 independent sums and 24 factor d_{j^+}, d_{j^-} and d_J, hence

$$\delta\mu \approx \sum_{6\times} \Lambda^{24}\Lambda^{-24} \approx \Lambda^6 \; , \tag{40}$$

which coincides with the results of [19, 9].

This power counting argument can be used to also derive for instance the degree of divergence of the graph $\mathcal{G}$ for the BF model with $SU(2)$ group ($\gamma = 1$). In this case the $-$ variables are absent and we have $n = 20$, $k = 5$ and we recover the well known scaling

$$\sum_{6\times} \Lambda^{18} \frac{1}{\sqrt{\Lambda}^{3\times 5}} \frac{1}{\sqrt{\Lambda}^{15}} = \Lambda^9 \; . \tag{41}$$

For an arbitrary graph the saddle point analysis becomes more involved, and the scaling is influenced both by the position of the saddle point in the θ space and by the presence of degenerate directions. A precise analysis is in progress [20].

As a final observation, note that the Barett Crane model $\gamma \to \infty$ has exactly the same divergences as the EPRL/FK model. We expect however that the sub leading divergences can be subtracted (in some "wave function" renormalization) only for the EPRL/FK models, leading to a non trivial flow of the Immirzi parameter.

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Appendix

A $SU(2)$ coherent states

An element g of $SU(2)$ writes $g = e^{i\frac{\theta}{2}\vec{k}\cdot\vec{\sigma}}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. In this parametrization the Haar measure on $SU(2)$ is

$$\int_{SU(2)} d\mu(g) = \frac{1}{2\pi}\int_0^{4\pi} d\theta \sin^2\frac{\theta}{2}\int_{S^2} dk \; . \tag{A.1}$$

Alternatively, elements of $SU(2)$ can be parametrized by Euler angles (in $z-y-z$ order)

$$g = e^{-i\alpha\sigma_z}\, e^{-i\beta\sigma_y} e^{-i\gamma\sigma_z} \; , \tag{A.2}$$

representing the rotation of angle γ around the direction

$$\vec{n} = (\sin\beta\, \cos\alpha,\ \sin\beta\, \sin\alpha,\ \cos\beta) \; , \tag{A.3}$$

and the Haar measure writes in terms of Euler angles as

$$\int_{SU(2)} d\mu(g) = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \int_{S^2} dn \; , \tag{A.4}$$

where we use the normalized measure on the sphere S^2

$$\int_{S^2} dn = \frac{1}{4\pi}\int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\alpha. \tag{A.5}$$

In the spin j representation space of $SU(2)$, $H_j = \{|j,m\rangle,\ |m| \leq j\}$, the Wigner matrix representing g writes in Euler angles

$$D^j_{pq}(g) = \langle j,p|g^j|j,q\rangle = e^{-i\alpha p} d^j_{pq}(\beta) e^{-i\gamma q} \; . \tag{A.6}$$

The coherent states on $SU(2)$ [24] are indexed by a vector $\vec{n}$

$$|j,n\rangle \equiv \sum_p D^j_{pj}(\alpha,\beta,0)|j,p\rangle \; . \tag{A.7}$$

Note that in the definition of the coherent states one uses Wigner matrices with γ, the third Euler angle, set to zero. When dealing with coherent states

one needs to reestablish the dependence of this Euler angles and transform integrals over the vector $\vec{n}$ into integrals over the $SU(2)$ group. Consider for instance the integral

$$\begin{aligned} d_j \int dn\, |j,n\rangle\langle j,n| \;\; &= \;\; \frac{d_j}{4\pi}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \\ &\quad \sum_{p,s} D^j_{pj}(\alpha,\beta,0)|j,p\rangle\overline{D^j_{sj}(\alpha,\beta,0)}\langle j,s| \; . \end{aligned} \tag{A.8}$$

We first add an extra normalized integral over a fictitious variable, χ,

$$\begin{aligned} &\frac{d_j}{4\pi}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \, \frac{1}{2\pi}\int_0^{2\pi} d\chi \\ &\sum_{p,s} D^j_{pj}(\phi,\psi,0)|j,p\rangle e^{-i\chi j} e^{i\chi j}\overline{D^j_{sj}(\phi,\psi,0)}\langle j,s| . \end{aligned} \tag{A.9}$$

But, due to eq. (A.6), $D^j_{pj}(\phi,\psi,0)e^{-i\chi j} = D^j_{pj}(\phi,\psi,\chi)$. Moreover, the integrals over α, β and χ reproduce the Haar measure on $SU(2)$, hence eq. (A.8) becomes

$$d_j \int dg \sum_{p,s} D^j_{pj}(\phi,\psi,\chi)\overline{D^j_{sj}(\phi,\psi,\chi)}|j,p\rangle\langle j,s| , \tag{A.10}$$

and using the orthogonality of the Wigner matrices

$$\int dg\; D^j_{pj}(g)\overline{D^j_{sj}(g)} = \frac{1}{d_j}\,\delta_{p,s} \tag{A.11}$$

one concludes that the coherent states yield a resolution of the identity

$$d_j \int dn\, |j,n\rangle\langle j,n| = \sum_p |j,p\rangle\langle j,p| = \mathbb{I}_j \; . \tag{A.12}$$

B Evaluation of $\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)$

In this appendix we compute the integral $\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)$ of eq. (10).

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2) \quad = \int dndk\; \langle\vec{j},\vec{m}_1|\Big(|j^+,n\rangle\otimes|j^-,n\rangle\Big)$$

$$\langle j^+ + j^-, n | j^+ + j^-, k\rangle \Big(\langle j^+, k| \otimes \langle j^-, k| \Big) |\vec{j}, \vec{m}_2\rangle \;. \quad \text{(B.1)}$$

Using the definition of coherent states eq. (A.7) and inserting judiciously phases in the new fictitious variables (generalizing straightforward the manipulation in appendix A) eq. (B.1) writes

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) = \quad & \sum_r \int dg dg' \, D^{j^+}_{m_1^+ j^+}(g) D^{j^-}_{m_1^- j^-}(g) \overline{D^{j^+ + j^-}_{r (j^+ + j^-)}(g)} \\ & D^{j^+ + j^-}_{r (j^+ + j^-)}(g') \overline{D^{j^+}_{m_2^+ j^+}(g')} \; \overline{D^{j^-}_{m_2^- j^-}(g')} \;. \end{aligned} \quad \text{(B.2)}$$

Under hermitian and complex conjugation the Wigner matrices satisfy the relations $D^j_{mn}(g^{-1}) = \overline{D^j_{nm}(g)} = (-1)^{n-m} D^j_{-n-m}(g)$, thus

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) \;\; = \;\; & \sum_r (-1)^{r - j^+ - j^- + m_2^+ - j^+ + m_2^- - j^-} \\ & \int dg D^{j^+}_{m_1^+ j^+}(g) D^{j^-}_{m_1^- j^-}(g) D^{j^+ + j^-}_{-r - (j^+ + j^-)}(g) \\ & \int dg' D^{j^+ + j^-}_{r (j^+ + j^-)}(g') D^{j^+}_{-m_2^+ - j^+}(g') D^{j^-}_{-m_2^- - j^-}(g') \;. \end{aligned} \quad \text{(B.3)}$$

The group integrals of products of three Wigner matrices compute in terms of Wigner 3j symbols [25]

$$\begin{aligned} & \int dg \; D^{j_1}_{m_1 n_1}(g) \, D^{j_2}_{m_2 n_2}(g) \, D^J_{M M'}(g) \\ & = \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M' \end{pmatrix} \;, \end{aligned} \quad \text{(B.4)}$$

thus

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) \;\; = \;\; & (-1)^{m_2^+ + m_2^- - 2(j^+ + j^-)} \sum_r (-1)^r \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_1^+ & m_1^- & -r \end{pmatrix} \\ & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ j^+ & j^- & -(j^+ + j^-) \end{pmatrix} \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ -m_2^+ & -m_2^- & r \end{pmatrix} \\ & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ -j^+ & -j^- & j^+ + j^- \end{pmatrix} \;, \end{aligned} \quad \text{(B.5)}$$

which writes using the symmetry properties of the $3j$ symbols

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) \;\; = \;\; & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ j^+ & j^- & -(j^+ + j^-) \end{pmatrix}^2 \sum_r (-)^{r + m_2^+ + m_2^-} \qquad \text{(B.6)} \\ & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_1^+ & m_1^- & -r \end{pmatrix} \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ -m_2^+ & -m_2^- & r \end{pmatrix} \;. \end{aligned}$$

Taking into account the evaluation of particular $3j$ symbols

$$\begin{pmatrix} j^+ & j^- & j^+ + j^- \\ j^+ & j^- & -(j^+ + j^-) \end{pmatrix} = \frac{(-1)^{2j^+}}{\sqrt{2(j^+ + j^-) + 1}} \,, \tag{B.7}$$

we get

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) = \frac{1}{d_{j^+ + j^-}} \quad & \sum_r (-)^{r + m_2^+ + m_2^-} \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_1^+ & m_1^- & -r \end{pmatrix} \\ & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ -m_2^+ & -m_2^- & r \end{pmatrix} . \end{aligned} \tag{B.8}$$

Also note that, according to [25],

$$\begin{aligned} & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ -m_2^+ & -m_2^- & r \end{pmatrix} = \frac{(-1)^{j^+ - j^- - r}}{\sqrt{d_{j^+ + j^-}}} \sqrt{\frac{(2j^+)!(2j^-)!}{(2j^+ + 2j^-)!}} \\ & \sqrt{\frac{(j^+ + j^- + r)!\,(j^+ + j^- - r)!}{(j^+ + m_2^+)!(j^+ - m_2^+)!(j^- + m_2^-)!(j^- - m_2^-)!}} \\ & = (-1)^{-2r} \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_2^+ & m_2^- & -r \end{pmatrix} . \end{aligned} \tag{B.9}$$

By the selection rules, the $3j$ symbol is zero unless $m_2^+ + m_2^- - r = 0$, hence we finally get

$$\begin{aligned} \mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) = \frac{1}{d_{j^+ + j^-}} \quad & \sum_r \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_1^+ & m_1^- & -r \end{pmatrix} \\ & \begin{pmatrix} j^+ & j^- & j^+ + j^- \\ m_2^+ & m_2^- & -r \end{pmatrix} , \end{aligned} \tag{B.10}$$

which can be rewritten as

$$\mathcal{I}(\vec{j}, \vec{m}_1, \vec{m}_2) = \frac{1}{d_{j^+ + j^-}} \sum_r \int dh \; D^{j^+}_{m_1^+, m_2^+}(h) D^{j^-}_{m_1^-, m_2^-}(h) D^{j^+ + j^-}_{-r, -r}(h) \; . \tag{B.11}$$

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